

Driven Random Field Ising Model: some exactly solved examples in threshold activated kinetics

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The random field Ising Model (RFIM) driven by a slowly varying uniform external field at zero temperature provides a caricature of several threshold activated systems. In this model, the non-equilibrium response of the system can be obtained analytically in one dimension as well as on a Bethe lattice if the initial state of the system has all spins aligned parallel to each other. We consider ferromagnetic as well as anti-ferromagnetic interactions. The ferromagnetic model exhibits avalanches and non-equilibrium critical behavior. The anti-ferromagnetic model is marked by the absence of these features. The ferromagnetic model is Abelian, and the anti-ferromagnetic model is non-Abelian. Theoretical approaches based on the probabilistic method are discussed in the two cases, and illustrated by deriving some basic results.

I. INTRODUCTION

A wide variety of physical systems exhibit threshold activated dynamics, i.e. they yield to an applied force only if the force has reached a threshold. For example, a book can be pushed across a table only if the applied force exceeds the threshold of static friction. More complex examples involve an infinite array of thresholds. Here we take up an example from the field of magnetism. A commercial magnet is typically a two phase material with fine particles of one phase embedded in the other phase. The precipitation is carried out in a magnetic field, and the particles are oriented with their long axis parallel to the field direction. After the material is set in a solid matrix, the needle like magnetic particles can not orient freely. Instead, the magnetic domains have to re-orient themselves if the external field is reversed. The orientation of magnetic domains in fine elongated particles requires a threshold field. The magnetization of the material in a smoothly increasing applied field may look smooth on a macroscopic scale but on a microscopic scale, it is made of steps of irregular widths and heights. This is known as Barkhausen noise. Thus, the trajectory of magnetization in an applied field is determined by a threshold activated process. When the applied field reaches a threshold, the magnetization jumps in a vertical step. Between two neighboring thresholds of the applied field, the magnetization remains constant. This marks the width of a step. The heights and widths of neighboring steps look apparently irregular, but the spectrum of the Barkhausen noise on a full hysteresis loop shows a power-law structure. The zero temperature dynamics of the ferromagnetic random field Ising model [1] provides a good caricature of this noise. The model has been solved exactly on a Bethe lattice, and the power-laws of the Barkhausen noise as well as some other features of the hysteresis loop have been obtained analytically [2, 3, 4, 5, 6, 7]. In the following we describe the model, and illustrate the method of its solution by calculating the hysteresis loop as the applied field is cycled from $-\infty$ to ∞ and back. Our method is applicable only if the initial state has all spins parallel to each other. Therefore, we start with the applied field equal to $-\infty$ when all spins are aligned along with the field. One can also calculate minor hysteresis loops within major hysteresis loop by reversing the applied field at arbitrary points on the major loop. It is also possible to calculate the distribution of avalanches (Barkhausen noise) on any part of the hysteresis loop. However, we do not go into these details here, and refer the reader to the literature.

We shall also discuss hysteresis in the anti-ferromagnetic random field Ising model at zero temperature [8]. Hysteresis in the anti-ferromagnetic RFIM is qualitatively different from that in the ferromagnetic RFIM because it does not show Barkhausen noise. On account of the anti-ferromagnetic interactions, a spin turning up in an increasing field blocks its nearest neighbors from turning up. Thus there is no avalanche of up-turned spins. Irreversibility or hysteresis in the ferromagnetic model comes from avalanches. A small increment in the applied field causes several spins to turn up in the ferromagnetic model, but reversing the increment does not cause all the spins to turn down. In the anti-ferromagnetic model, there are no avalanches. So we might think that there should be no hysteresis in the anti-ferromagnetic model. It is true that if only one spin turned up at each threshold, there would be no hysteresis in the anti-ferromagnetic model i.e. the area of the hysteresis loop would go to zero. However, there is a small amount

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of hysteresis in the anti-ferromagnetic model that comes from the following effect. A spin that turns up, occasionally causes its nearest neighbor (that had turned up earlier) to turn down. In other words, as the applied field increases from $-\infty$ to $+\infty$, a small fractions of sites flip three times, first up, then down, and finally up again.

The lack of avalanches in the anti-ferromagnetic model also sets it apart from the ferromagnetic model in another aspect. Consider a stable state of the model in a given applied field, and imagine increasing the applied field slightly. Suppose there are two neighboring sites that have the potential to turn up in the increased field. In the ferromagnetic case, we can turn up any of these spins first, and the other spin would also turn up subsequently. The exact sequence in which the spins are relaxed in the ferromagnetic model is not important. This property of the ferromagnetic model is called the Abelian property. The anti-ferromagnetic model is non-Abelian. If two sites can both turn up at a field, it is important to decide which is to be turned up first i.e. which site has a larger net field. Once it is turned up the other site is blocked from turning up until a higher applied field. This constant revision of the threshold distribution by the dynamics of the anti-ferromagnetic RFIM is the key difficulty in an exact solution of the model. So far the anti-ferromagnetic model has been solved exactly in one dimension only.

II. THE MODEL

Consider a lattice of N sites. Each site is labeled by an integer $i = 1$ to N , and carries an Ising spin S_i ($S_i = \pm 1$), a quenched random magnetic field h_i , and an externally applied uniform field h . The quenched fields $\{h_i\}$ are independent identically distributed random variables with a continuous probability distribution $\phi(h_i)$. There is an interaction J between nearest neighbor spins that can be ferromagnetic ($J > 0$) or anti-ferromagnetic ($J < 0$). The Hamiltonian of the system is

$$H = -J \sum_{i,j} S_i S_j - \sum_i h_i S_i - h \sum_i S_i \quad (1)$$

The spin S_i experiences a net field f_i on it that is given by,

$$f_i = J \sum_j S_j + h_i + h \quad (2)$$

The Glauber dynamics of the system at temperature T is specified by the rate R_i at which a spin S_i flips to $-S_i$

$$R_i = \frac{1}{\tau} [1 - S_i \tanh\{f_i/(k_B T)\}], \quad (3)$$

Here τ sets the basic time scale for the relaxation of individual spins. The energy of the spin S_i is equal to $-f_i S_i$. If f_i and S_i have the same sign, we say that the spin is aligned along the net field at its site. The energy of a spin is the lowest if it is aligned along the net field at its sight. We are interested in the dynamics of the model at zero temperature ($T = 0$), and on time scales much larger than τ . In this limit, the dynamics simplifies to the following rule: choose a spin at random, and flip it only if it is not aligned along the net field at its site. Repeat the process till all spins are aligned along the net fields at their respective sites.

We apply the above dynamics to calculate hysteresis in the system when it is driven by a uniform external field $h(t) = -h_0 \cos \omega t$. We work in the limit $h_0 \rightarrow \infty$, and $\omega \rightarrow 0$. In other words, we start with a sufficiently large negative applied field h_0 so that the stable configuration has all spins down ($S_i = -1$, for all i), and increase the field slowly. At some value of $h(t)$, the net field f_i at some site i will become positive, and S_i would flip up. This changes the net field at the neighbors, and in the ferromagnetic case some of the neighbors may flip up, and so on, causing an avalanche of flipped spins. We keep the applied field fixed during an avalanche. After the avalanche has stopped and a stable state has been reached, we calculate the magnetization per site $m(h)$;

$$m(h) = \frac{1}{N} \sum_i S_i \quad (4)$$

Then we raise the applied field slightly till the next avalanche occurs, and calculate the magnetization again after that avalanche has stopped. This process is continued until the applied field is sufficiently large and positive (h_0), and all spins are up. The magnetization trajectory $m_l(h)$ determined in this way gives the lower half of the hysteresis loop

in the limit of zero driving frequency. The upper half of the hysteresis loop $m_u(h)$ is similarly obtained by decreasing the applied field from h_0 to $-h_0$. This is related to $m_l(h)$ by symmetry, $m_u(h) = -m_l(-h)$. Note that the area of the hysteresis loop in this model does not go to zero even if the driving frequency goes to zero because the limit $T \rightarrow 0$ has been taken before the limit $\omega \rightarrow 0$.

The hysteresis loop obtained by the zero temperature dynamics of the random field Ising model described above is a good model of low temperature hysteresis seen experimentally in several systems. In experimental systems, the magnetization states on the two halves of the hysteresis loop are metastable states with a life time much longer than any other time scale of practical interest. Our model replaces the metastable states by fixed point states, but the fixed points correspond to the metastable states in the sense that they represent local minima of energy at a particular applied field. In this context, it is useful to consider the time scales of interest in physical systems, and the appropriateness of the zero-temperature dynamics as a model. In a typical hysteresis experiment, there are at least four time scales: (i) typical time τ that an individual spin takes to relax, (ii) time τ_1 that the system takes to relax to a metastable state, (iii) time τ_2 over which the applied field changes, and (iv) life time τ_3 of the metastable state. In complex systems, τ_1 , τ_2 , and τ_3 may each contain an entire spectrum of time scales. In physical systems relevant to our model, the shortest time is τ which may be taken to be unity to set the scale. The next larger time is $\tau_1 = \nu \times \tau$, where ν is the number of iterations of the dynamics to reach a fixed point. The applied field is assumed to vary very slowly (driving frequency goes to zero!) so that $\tau_2 \gg \tau_1$. Specifically, it means that the applied field is held constant during the relaxation of the system. The time τ_3 is infinite. Thus, the present model is applicable if $\tau_3 \gg \tau_2 \gg \tau_1 \gg \tau$. These conditions fit a wide class of complex magnetic materials that have a large number of metastable states separated from each other by barriers much larger than the available thermal energy.

III. FERROMAGNETIC MODEL ON A BETHE LATTICE

A Bethe lattice is an infinite-size branching tree of coordination number z . We choose a site at random (call it the central site), and ask for the probability that this site is up when the applied field has been increased from $-\infty$ to h . Each nearest neighbor of the central site forms the vertex of an infinite sub-tree. If the central site were to be deleted, the lattice would break up into z disjointed pieces. This means that the spin-flip dynamics on z sub-trees is independent of each other as long as the central site does not flip up from its initial state. The Abelian nature of the ferromagnetic model allows us to relax the sites in any order of our choice. We choose to relax the central site after its neighbors have been relaxed. Let $P^*(h)$ denote the conditional probability that a nearest neighbor of the central site is up at field h before the central site is relaxed (i.e. given that the central site is down). Then the probability that the central site is up at h is given by,

$$p(h) = \sum_{n=0}^z \binom{z}{n} [P^*(h)]^n [1 - P^*(h)]^{z-n} p_n(h) \quad (5)$$

The last factor, $p_n(h)$, gives the probability that a spin with n nearest neighbors up, and $z - n$ neighbors down has sufficient random field to flip it up at an applied field h .

$$p_n(h) = \int_{(z-2n)J-h}^{\infty} \phi(h_i) dh_i \quad (6)$$

The magnetization per site is given by $m(h) = 2p(h) - 1$. The calculation of P^* proceeds as follows. Consider a nearest neighbor of the central site and its sub-tree. Let the nearest neighbor i.e. the vertex of the sub-tree be n steps away from the boundary of the tree (the limit $n \rightarrow \infty$ will follow). The sites on the boundary of the sub-tree have only one nearest neighbor situated at a height of one step from the boundary. We relax the boundary sites first holding sites at height-1 down. Then we relax sites at height-1, holding sites at height-2 down., and continue this process. The conditional probability P^m that a site at height- m is up given that its nearest neighbor at height $m + 1$ is held down, satisfies the recursion relation,

$$P^m(h) = \sum_{n=0}^{z-1} \binom{z-1}{n} [P^{m-1}(h)]^n [1 - P^{m-1}(h)]^{z-1-n} p_n(h). \quad (7)$$

The above equation is understood as follows. A site at height m has one neighbor at height $m + 1$, and $z - 1$ neighbors at a height $m - 1$. The neighbor at height $m + 1$ is held down while the site at m is relaxed. Each neighbor

at height $m - 1$ can be independently up with probability $P^{m-1}(h)$ before the site at height m is relaxed. The last factor $p_n(h)$ gives the probability that the site at height m flips up when relaxed. As, $m \rightarrow \infty$, the above equation iterates to the fixed point value $P^*(h)$.

If the applied field is reversed before completing the lower half of the major hysteresis loop, we generate what is known as a minor hysteresis loop. First reversal of the field generates the upper half of the minor loop, and a second reversal generates the lower half. When the field on second reversal reaches the point where the first reversal was made, the lower half of the minor loop meets the starting point of the upper half. That is, the minor loop closes upon itself at the point it started. This property of the RFIM is known as return point memory.

The analytic calculation of minor loop is more difficult technically than that of the major loop. Consider the upper half of the minor loop. Suppose the applied field is reversed from h to h' ($h' \leq h$). We want to know the probability that an arbitrary site i which was up at h turns down at h' . It is necessary to know how many nearest neighbors of site i are up at h' . But this is not enough. We also need to know how many of the up neighbors were up before site i turned up, and how many turned up after site i . When site i turns up, the field on each nearest neighbor increases by an amount $2J$. It does not affect the neighbors which are already up but some down neighbors may turn up as a result of the increased field. For each down neighbor that turns up after site i , the field on site i increases by an amount $2J$. Even if one down neighbor turns up, site i will not turn down if the applied field that causes it to turn up is rolled back infinitesimally. It has to be rolled back sufficiently so that the neighbor which turned up after site i turns back down again. Site i can not turn down until all neighbors which turned up after it have turned down. This is the origin of hysteresis in the model, and also the difficulty in the calculation of the minor loop. We have to calculate the probability $D^*(h')$ that a nearest neighbor of site i that was down before site i turned up is down again at h' . $D^*(h')$ is determined by the equation,

$$D^*(h') = \sum_{n=0}^{z-1} \binom{z-1}{n} [P_l^*(h)]^n [1 - P_l^*(h)]^{z-1-n} [1 - p_{n+1}(h)] \\ + \sum_{n=0}^{z-1} \binom{z-1}{n} [P_l^*(h)]^n [D^*(h')]^{z-1-n} [p_{n+1}(h) - p_{n+1}(h')] \quad (8)$$

Given a site i that is up at h , the first sum above gives the conditional probability that a nearest neighbor of site i remains down at h after site i has turned up. The second sum takes into account the situation that the nearest neighbor in question turns up at h after site i turns up but turns down at h' .

The fraction of up sites which turn down at h' is given by,

$$q'(h') = \sum_{n=0}^z \binom{z}{n} [P_l^*(h)]^n [D^*(h')]^{z-n} [p_n(h) - p_n(h')] \quad (9)$$

The magnetization on the upper return loop is given by,

$$m'(h') = 2[p(h) - q'(h')] - 1 \quad (10)$$

We reverse the field h' to h'' ($h'' > h'$) to trace the lower half of the return loop. The magnetization on the lower half of the return loop may be written as,

$$m''(h'') = 2[p(h) - q'(h') + p''(h'')] - 1 \quad (11)$$

where $p''(h'')$ is the probability that an arbitrary site i which turned up at h and turned down at h' , turns up again at h'' .

$$p''(h'') = \sum_{n=0}^z \binom{z}{n} [U^*(h'')]^n [D^*(h')]^{z-n} [p_n(h'') - p_n(h')] \quad (12)$$

Here $U^*(h'')$ is the conditional probability that a nearest neighbor of a site i turns up before site i turns up on the lower return loop. It is determined by the equation,

$$\begin{aligned}
U^*(h'') = & P^*(h) - \sum_{n=0}^{z-1} \binom{z-1}{n} [P_l^*(h)]^n [D^*(h')]^{z-1-n} [p_n(h) - p_n(h')] \\
& + \sum_{n=0}^{z-1} \binom{z-1}{n} [U^*(h'')]^n [D^*(h')]^{z-1-n} [p_n(h'') - p_n(h')]
\end{aligned} \tag{13}$$

The rationale behind equation (13) is similar to the one behind equation (8). Given that a site i is down at h' , the first two terms account for the probability that a nearest neighbor of site i is up at $h'' \geq h'$. Note that the neighbor in question must have been up at h in order to be up at h' , and if it is already up at h' then it will remain up on the entire lower half of the return loop, i.e. at $h'' \geq h'$. The third term gives the probability that the neighboring site was down at h' , but turned up on the lower return loop before site i turned up. It can be verified that the lower return loop meets the lower major loop at $h'' = h$ and merges with it for $h'' > h$ as may be expected on account of the return point memory.

The analytic results on the Bethe lattice are obtained by taking the infinite-size limit of a branching tree. Fixed points of recursion relations have the effect of eliminating surface effects. It is rather difficult to eliminate surface effects in numerical calculations on branching trees because most of the sites on a finite tree are on the boundary or close to the boundary. We therefore perform the numerical simulations of the model on random graphs of coordination z . A random graph of N sites has no surface, but the price we pay is that it has some loops. However, for almost all sites in the graph, the local connectivity up to a distance of $\log_{(z-1)} N$ is similar to the one in the deep interior of the branching tree. Therefore, simulation on a random graph is a very efficient method of subtracting the surface effects on the corresponding finite branching tree. Figure 1 shows a comparison between the theoretical expression and the numerical simulation for $z=4$, and $\sigma = 1.7$. The agreement is quite remarkable considering the simulations are performed on a random graph, and the theoretical result is obtained by taking the infinite size limit of a branching tree.

The analysis presented above shows that a reversal of the applied field by an amount $2J$ at any point on the lower half of the major hysteresis loop brings the system on the upper half of the loop. This result provides an interesting possibility for measuring the exchange interaction J in a hysteresis experiment. Another point that is brought out by the above analysis is the following. Hysteresis on a Bethe lattice of coordination $z > 3$ is qualitatively different from the case $z=2$, and $z=3$. For $z \geq 4$, there is a critical value of σ that characterizes the Gaussian random field distribution. If σ is less than the critical value σ_c , the magnetization in increasing field has a macroscopic first-order jump at an applied field $h_c > J$. As σ increases to σ_c , h_c decreases to J , and the first-order jump in magnetization reduces to zero. The system shows non-equilibrium critical behavior at $h = h_c$, and $\sigma = \sigma_c$. For $z=4$, $\sigma_c = 1.78$ approximately. Bethe lattices with $z=2$, and $z=3$ do not show a macroscopic jump in the magnetization or critical behavior for any value of σ . Perhaps this dependence of the non-equilibrium critical behavior on the coordination number of the lattice has a more general validity going beyond the Bethe lattice. Numerical simulations and theoretical arguments on several periodic lattices embedded in two and three-dimensional space show that hysteresis on periodic lattices with $z \geq 4$ is qualitatively different from that on lattices with $z < 4$. Although there are some similarities between Bethe lattices and periodic lattices of the same coordination number, there are differences as well. The differences are related to the bootstrap percolation instability on some periodic lattices [9].

The method of calculating the minor loop described above may also be extended to obtain a series of minor loops nested within the minor loop obtained above. The key point is that whenever the applied field is reversed, a site i may flip only after all neighbors of site i which flipped in the wake of site i (on the immediately preceding sector) have flipped back. The neighbors of site i which remained firm after site i flipped do not yield before site i has flipped. We have obtained above expression for the return loop when the applied field is reversed from $h_{ext} = h$ on the lower major loop to $h_{ext} = h'$ ($h - 2J \leq h' \leq h$), and reversed again from $h_{ext} = h'$ to $h_{ext} = h''$ ($h'' \leq h$). When the applied field is reversed a third time from h'' to h''' ($h''' < h''$), expressions for the magnetization on the nested return loop follow the same structure as the one on the trajectory from h to h' . Qualitatively, the role of P^* on the first leg (h to h') is taken up by U^* on the third leg (h'' to h''') of the nested return loop.

It was noted earlier [8] that the major hysteresis loops for the RFIM on Bethe lattices of coordination number z have discontinuities for certain distributions of the random field. For a Gaussian distribution of the random field with mean value zero and variance σ^2 , a discontinuity in each half of the major hysteresis loop occurs for values of σ smaller than a critical value, and for $z \geq 4$. There is no discontinuity for a Gaussian distribution on lattices with $z = 2$, and $z = 3$. These considerations apply to minor loops as well. The set of equations determining the minor loops are polynomials of degree z , and stable solutions of these equations in a continuously changing applied field may change discontinuously. At a critical applied field where the major loop has a jump discontinuity, we have the possibility of two minor loops depending upon the magnetization of the state which is used to generate the minor loop.

IV. ANTI-FERROMAGNETIC MODEL IN ONE DIMENSION

As mentioned earlier, the zero temperature dynamics of the anti-ferromagnetic random field Ising model is non-Abelian. The stable configuration in an applied field h depends on the order in which the sites are relaxed, in addition to the hysteretic dependence on the initial state of the system. As we raise the applied field from $h = -\infty$ very slowly, spins turn up in a sequence in which the net fields at their sites vanish. When a spin turns up, the net field on its neighbors is decreased by an amount $2|J|$. This blocks the neighbors from turning up until the applied field has increased by at least an amount $2|J|$.

In our model, the rate at which the applied field is increased does not have any fundamental significance. We can expose the initial state (with all spins down) directly in the field h , and relax the spins in the order they would have followed if the field were increased infinitely slowly. Let $p_0(h)$ denote the probability that in an applied field h , a spin with random field h_i has the potential to turn up if none of its neighbors are up, i.e. $p_0(h)$ is the probability that the quantity $[h_i + 2|J| + h]$ is greater than zero before the relaxation process is executed. During relaxation, the fraction of sites that actually turn up may be much less than $p_0(h)$, especially for large $p_0(h)$. The reason is that the potential sites have to be relaxed sequentially in the order of decreasing net field on them. A potential site that is adjacent to another potential site with a higher field gets blocked by it. It takes a relatively larger applied field to turn up a spin at a blocked site.

We consider the stable state of a one dimensional model in an applied field h . If h is not too high, the stable state may be punctuated by pairs of adjacent spins along the chain that have never flipped from their initial state. These pairs of down spins (to be called doublets below) serve to shield the dynamics of the chain on one side of the pair from that on the other side. The evolution on a finite segment of the chain between two doublets is independent of the rest of the chain. It is therefore of interest to calculate the density of doublets in a stable chain. If $P_{\downarrow\downarrow}$ denotes the conditional probability that a spin is down in a stable state given that one of its nearest neighbors is down, then $P_{\downarrow\downarrow}^2$ is the probability per site of observing a doublet on the chain.

Consider a string of $n+1$ sites embedded in the chain. Number the first site at the left end of the string as 0, and the following sites as 1, 2, ..., n . Given that site-0 is down, we wish to calculate the conditional probability $P_{\downarrow\downarrow}[0 \downarrow: 1 \downarrow]$ that site-1 is down. Site-1 may be down because it does not have the potential to turn up, or it may have the potential to turn up but may be blocked by site-2. If m nearest neighbors of a spin are up just before that spin turns up, we say that it has turned up under a p_m -process ($m=0,1,2$). If a spin does not have the potential to turn up under a p_0 -process, then it does not have the potential to turn up at all. The probability that site-1 does not have the potential to turn up is equal to $1 - p_0(h)$. Thus,

$$P_{\downarrow\downarrow}[0 \downarrow: 1 \downarrow] = 1 - p_0(h); (2|J| + h_1 + h < 0). \quad (14)$$

Calculation of the probability that site-1 has the potential to turn up but is blocked by site-2 is relatively complicated. Our strategy is to calculate it in an approximation initially, and then incorporate corrections in our calculation to make it exact. The approximation we use is to assume that only p_0 -processes operate initially; p_1 and p_2 -processes are put on hold temporarily. In other words, only those spins turn up whose neighbors on both sides are down. If the spin at site-1 is blocked, then the spin at site-2 must be up. The probability that the spin at site-2 is up depends upon how the spins have turned up on the right of site-2. How far does the correlation extend? The correlation must end at the first occurrence of a doublet. But it may end earlier at a site that is the first site to flip up on the right of site-1. Suppose site- n is the first site to flip up. Then site-1 will be correlated to site- n if $h_n > h_{n-1} > h_{n-2} \dots h_2 > h_1$. Note that if the above chain of inequalities is not satisfied, we would get a doublet between site-1 and site- n contrary to our assumption. The probability of getting n potential sites with random fields ordered as above is $p_0^n/n!$. In this case, the next site to turn up after site- n will be site-($n-2$) followed by sites ($n-4$), ($n-6$), and so on. Site-1 will end up being down if n is even, and up if n is odd. Therefore the probability that site-1 will be down at the end of the relaxation process is given by,

$$P_{\downarrow\downarrow}^a[0 \downarrow: 1 \downarrow] = \sum_{n=2}^{\infty} \frac{[-p_0(h)]^n}{n!} (2|J| + h_1 + h > 0; p_1, p_2 \text{-processes on hold.})$$

We can rewrite the above equation as,

$$\begin{aligned} P_{\downarrow\downarrow}^a[0 \downarrow: 1 \downarrow] &= g(h); (2|J| + h_1 + h > 0; p_1, p_2 \text{-processes on hold.}); \\ g(h) &= \exp\{-p_0(h)\} - \{1 - p_0(h)\} \end{aligned} \quad (15)$$

Combining the preceding two equations, we get the conditional probability that the spin at site-1 is down whether site-1 is a potential site or not.

$$P_{\downarrow\downarrow}[0 \downarrow: 1 \downarrow] = \exp \{-p_0(h)\} \text{ (} p_1, p_2 \text{ -processes on hold.)} \quad (16)$$

Now we put in the corrections to get an exact result. If p_1 and p_2 -processes are allowed to proceed along with the p_0 -process, site-2 can turn up in two ways. It may turn up under a p_0 -process, or it may turn up under a p_1 -process. If site-2 turns up under a p_0 process, site-1 may turn up under a p_1 - process subsequently. This event decreases $P_{\downarrow\downarrow}[0 \downarrow: 1 \downarrow]$ below the value $\exp(-p_0)$ calculated above. On the other hand, if site-3 turns up under a p_0 -process , and subsequently if site-2 turns up under a p_1 -process, site-1 which would have otherwise turned up under a p_0 -process may stay down on account of being blocked by site-2. This event increases $P_{\downarrow\downarrow}[0 \downarrow: 1 \downarrow]$ above $\exp(-p_0)$. In the following we calculate the two correction terms mentioned above.

1. If the spin site-2 turns up under a p_0 -process, the probability that the spin at site-1 is down, and $h_1 + 2|J| + h > 0$, is equal to $g(h)$. If site-1 is to turn up after site-2, we must have $h_1 + h > 0$. The probability for this event is $g(h - 2|J|)$. The amount $g(h - 2|J|)$ has to be subtracted from $\exp(-p_0)$.
2. Consider the event that the spin at site-3 turns up under a p_0 -process, subsequently the spin at site-2 turns up under a p_1 -process, and the spin at site-1 stays down. If the spin at site-1 were to turn up as well, then we do not get any additional contribution to $P_{\downarrow\downarrow}[0 \downarrow: 1 \downarrow]$. The probability that the spin at site-1 remains down is given by,

$$P_{\downarrow\downarrow}^a[0 \downarrow: 1 \downarrow] = \int_{-h-2|J|}^{-h} \phi(h_1) g(-h_1 - 4|J|) dh_1 \quad (17)$$

The limits on integration take into account the fact that the random field h_1 at site-1 has to satisfy the inequalities $h_1 + 2|J| + h \geq 0$, and $h_1 + h < 0$. Also, we must have $h_2 \geq h_1 + 2|J|$ in order for site-2 to flip up under a p_1 -process before site-1 can flip under a p_0 -process. The probability for this is given by $g(-h_1 - 4|J|)$. The amount $P_{\downarrow\downarrow}^a[0 \downarrow: 1 \downarrow]$ is to be added to $\exp(-p_0)$.

Putting the three terms together, the conditional probability that site-1 is down given that site-0 is down is,

$$P_{\downarrow\downarrow}[0 \downarrow: 1 \downarrow] = \exp \{-p_0(h)\} - g(h - 2|J|) + P_{\downarrow\downarrow}^a[0 \downarrow: 1 \downarrow] \quad (18)$$

Cognizant readers may notice that we have considered the possibility of only one spin turning up under a p_1 -process. But several spins may turn up under a p_1 -process. Indeed, the quenched field at site-1 is the smallest of the set of quenched fields $\{h_n > h_{n-1} > h_{n-2} \dots h_2 > h_1\}$. Therefore, if the spin at site-1 can turn up under a p_1 -process then all other spins under consideration can also turn up under a p_1 -process. Suppose spin at site-4 turns up under a p_0 -process, and subsequently spins at site-3, and site-2 turn up under a p_1 -process. This would happen if $h_3 - h_2 > 2|J|$, and $h_2 - h_1 > 2|J|$. However, the resulting state in this case could be obtained by relaxing site-4 and site-2 under a p_0 -process, and then site-3 under a p_2 -process. Care has to be exercised when calculating the fraction of up spins in this way (some spins may turn down when their neighbor turns up), but it does not contribute any extra term to doublets. Similarly, the extra doublet that was considered above need not necessarily occur on site-0 and site-1. It can occur at any other position between site-1 and site-n, but the correction term $P_{\downarrow\downarrow}^a[0 \downarrow: 1 \downarrow]$ remains the same. The significance of site-1 in our discussion is only that its left neighbor at site-0 is held fixed in the down state.

In Figure (2) we show the calculated probability of observing doublets in a stable state of the anti-ferromagnetic chain at an applied field h for a Gaussian distribution of the random field with $\sigma=0.5$, and $\sigma=1$. In each case, result from an appropriate numerical simulation has been superimposed on the theoretical expression. The agreement between the theoretical expression and the corresponding numerical result is quite good. In fact, on the scale of figure (2) the two are indistinguishable from each other.

The calculation of doublets presented above allows us in principle to determine the modified a posteriori distribution of random fields on all spins that are down. We can therefore calculate the probability of any spin turning up under a p_0 , p_1 , or p_2 -process. Thus the magnetization at any applied field h may be calculated. The details however are rather tedious. For a rectangular distribution of the random field with mean zero and width 2Δ ($\Delta < |J|$), the reader may refer to reference [8]. The simplifying feature of the case $\Delta < |J|$ is that $(h_i - h_{i-1} < 2|J|)$ for any site i on the chain. Therefore no p_1 -process can take place before all p_0 -processes have been exhausted. Similarly no p_2 -process can take place before all p_1 -processes have been exhausted. The details of the calculation of magnetization for an unbounded distribution of the random field will be given elsewhere.

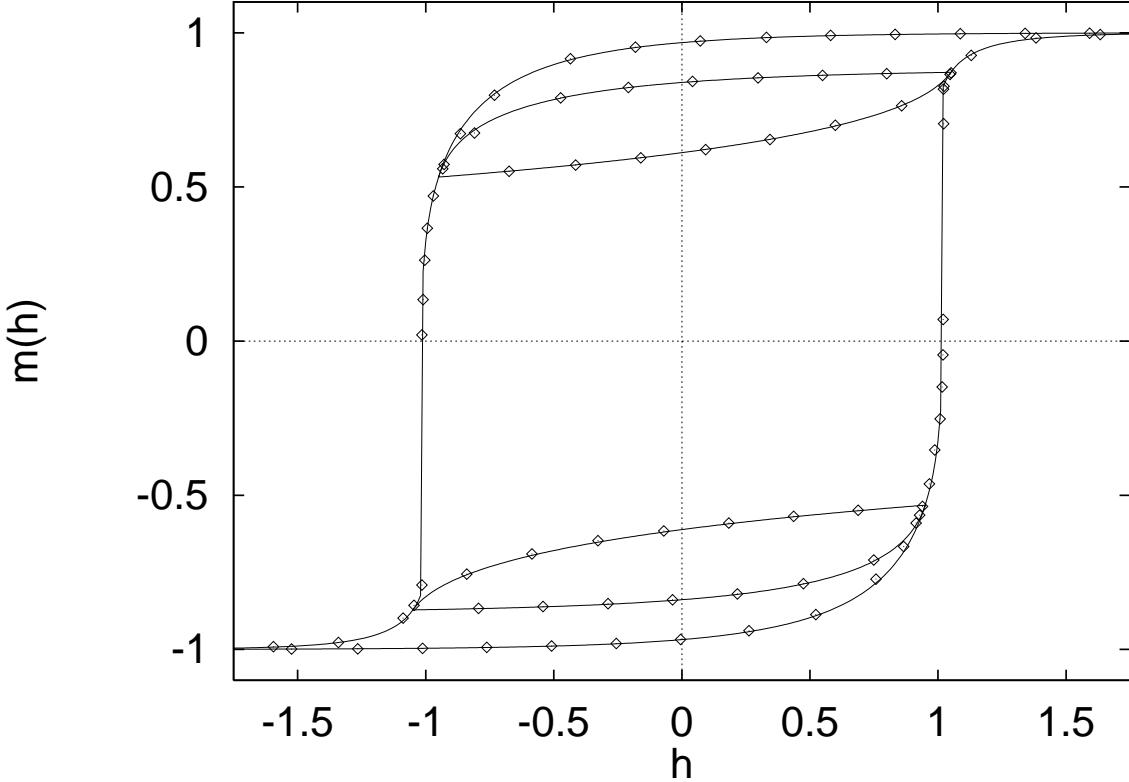


FIG. 1: Hysteresis in the ferromagnetic RFIM on a Bethe lattice with coordination number $z=4$, and a Gaussian distribution of random field with mean zero and $\sigma = 1.7$. Continuous lines show the theoretical result, and dots mark the results of numerical simulations. Notice the first order jumps in the magnetization at applied field $h \approx \pm 1$ ($J=1$) because $\sigma < \sigma_c$. Two minor loops within the major loop are obtained by reversing the increasing field at $h = .95$, and $h = 1.05$ respectively. The minor loops touch the upper half of the major loop when the field has been reversed by an amount $2 |J|$ (see text).

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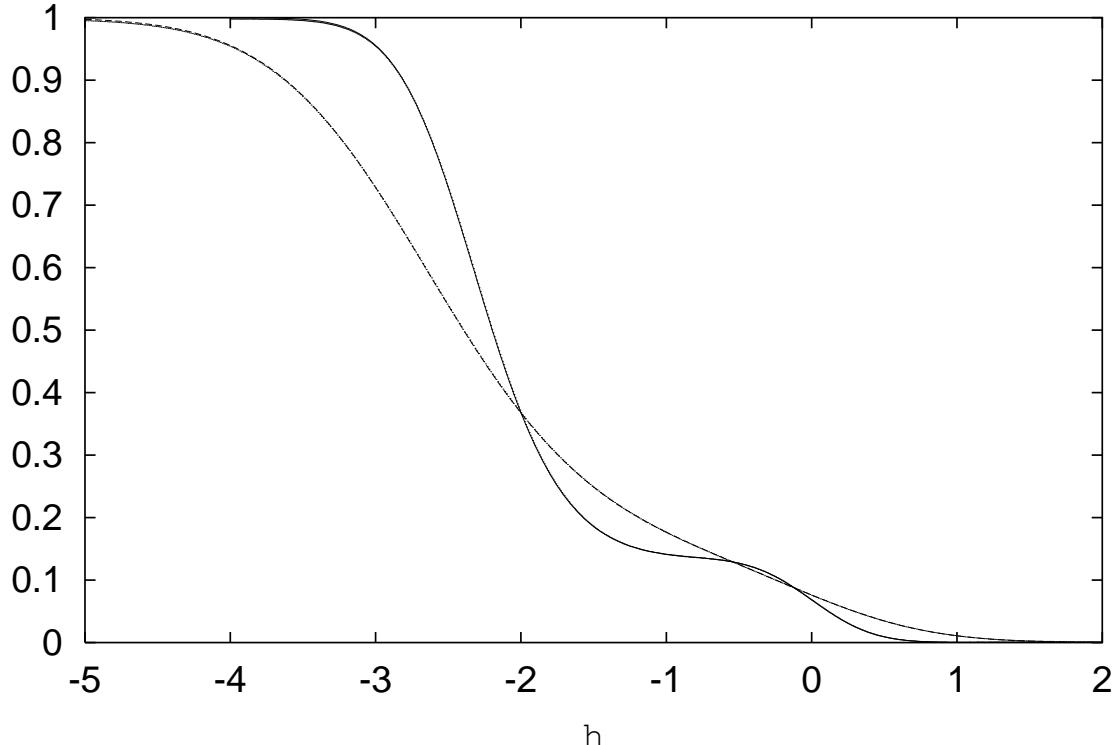


FIG. 2: Occurrence of doublets in a one dimensional anti-ferromagnetic RFIM. The y-axis shows the probability of occurrence of a doublet $P_{\downarrow\downarrow}(h)$ at an applied field h . The curve with a plateau like region in the middle is for $\sigma = .5|J|$, and the other curve is for $\sigma = |J|$. Results from numerical simulations have been superimposed on the theoretical curves.